Dynamics of Induced Surfaces in Four-Dimensional Euclidean Space

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Abstract

The Davey Stewartson hierarchy will be developed based on a set of three matrix differential operators. These equations will act as evolution equations for different types of surface deformation in Euclidean four space. The Weierstrass representation for surfaces will be developed and its uniqueness up to gauge transformations will be reviewed. Applications of the hierarchy will be given with regard to generating deformations of surfaces, and it will be shown that the Willmore functional is preserved under this kind of deformation.

1. INTRODUCTION.

Surfaces and the dynamics of surfaces play a very essential role in many areas of classical as well as quantum physics. Moreover, as far as the area of classical differential geometry is concerned, the theory of the immersion and deformations of surfaces has been the subject of intense research [**1**].

Domains of study, such as surface waves, deformation of membranes, dynamics of vortex sheets as well as certain problems in the area of hydrodynamics are related to the motion of boundaries which separate different regions. In particular, in the area of string theory, the action is related to the Polyakov integral over surfaces. Certain special classes of surfaces give important contributions to various types of physical quantities appearing in these theories and are of interest to consider.

Recently, Konopelchenko generalized the Weierstrass formulas to the case of generic surfaces in $\mathbb{R}^3$ [**2-3**]. These formulas can be used to study the global properties of surfaces in $\mathbb{R}^3$, as well as the integrable deformations of such surfaces. This latter aspect is perhaps one of the more important reasons for developing these kinds of techniques for inducing surfaces in higher dimensional spaces which include Minkowski type spaces as well as Euclidean spaces. These representations can then be used to study not only the geometry of surfaces, but the integrable deformations of such surfaces as well.

The generalization of the Weierstrass formulas to generic surfaces in $\mathbb{R}^3$ which was proposed by Konopelchenko consists of the linear system of Dirac equations

$$\partial\psi = p\varphi, \qquad \bar{\partial}\varphi = -p\psi, \tag{1.1}$$

where ψ and φ are complex-valued functions of z, $\bar{z} \in \mathbb{C}$ and $p(z,\bar{z})$ is a real-valued function. The derivative operators will be abbreviated to $\partial = \partial/\partial z$ and $\bar{\partial} = \partial/\partial\bar{z}$ throughout. Using solutions to (1.1), three real-valued functions $X^1(z,\bar{z})$, $X^2(z,\bar{z})$ and $X^3(z,\bar{z})$ are defined by the integrals

$$X^1 + iX^2 = i\int_\Gamma (\bar{\psi}^2\, dz' - \bar{\varphi}^2\, d\bar{z}'),$$

$$X^1 - iX^2 = i\int_\Gamma (\varphi^2\, dz' - \psi^2\, d\bar{z}'), \tag{1.2}$$

$$X^3 = -\int_\Gamma (\bar{\psi}\varphi\, dz' + \psi\bar{\varphi}\, d\bar{z}'),$$

where Γ is an arbitrary curve in $\mathbb{C}$. Regarding the $X^i(z, \bar{z})$ as coordinates in $\mathbb{R}^3$, (1.1) and (1.2) define a conformal immersion of a surface into $\mathbb{R}^3$. It has been shown that the generalized Weierstrass representation of Konopelchenko has several other links to areas of mathematical physics, in particular, it can be directly related to the $\mathbb{C}P^1$ nonlinear sigma model [**4**]. In fact, solutions of one system can be transformed into solutions of the other. Moreover both the generalized Weierstrass system and the nonlinear sigma model are completely integrable systems [**5**]. These types of representations are useful because in this context, deformations of surfaces can be discussed in a straightforward way. Konopelchenko made an important remark in this regard. If a surface is represented locally by the generalized Weierstrass representation, then based on the operator

$$L = \begin{pmatrix} \partial & -p \\ p & \bar{\partial} \end{pmatrix}$$

such that the potential p satisfies a particular evolution equation, the deformation in p obtained under the evolution equation induces a local deformation of a surface. The relevant integrable evolution equation considered here is the Novikov-Veselov equation, although other integrable systems could be considered. Taimanov [**6**] showed that these formulas for inducing surfaces in $\mathbb{R}^3$ describe all surfaces and that the modified Novikov-Vesolov equation deforms tori into tori preserving the Willmore functional. These types of application lend an important role to this type of representation. The analogous problems for surfaces in $\mathbb{R}^4$ show that this case is very different from the three-dimensional case. The main reason is that for tori in $\mathbb{R}^4$, each equation of the Davey-Stewartson (DS) hierarchy describes not one but infinitely many geometrically different soliton deformations, and is linked to the fact that the representation is not unique in this case.

To put it concisely, a surface in $\mathbb{R}^3$ is constructed from a single vector or spinor function ψ which is a lift of the Gauss map into nonvanishing spinors, such that it satisfies a Dirac equation, and the lift is defined up to a sign. On the other hand, a surface in $\mathbb{R}^4$ is obtained from two spinors ψ and φ which form again a lift of the Gauss map. However, in this case the lift is defined only up to a gauge transformation [**7-10**] given by e^f, where f is any smooth function.

It is the purpose in this paper to introduce the system of equations satisfied by these two spinors

in $\mathbb{R}^4$, and the corresponding equations for inducing the corresponding surface. The DS-hierarchy will be introduced and a mechanism for deforming these surfaces will be considered. In particular, a recipe will be given for obtaining the relevant DS equations pertaining to the first three elements of the hierarchy from a matrix system using symbolic manipulation [**11**]. Applications of these evolution equations will be given for deformation of surfaces, and some results related to the Weierstrass representation and its gauge invariance will be explored. Finally, it will be shown that the Willmore functional is preserved under this kind of deformation.

2. REPRESENTATION OF SURFACES AND THEIR DEFORMATIONS

An extension of the generalized Weierstrass representation to four-dimensional Euclidean space is based on a pair of spinors ψ and φ whose components can be regarded as two independent solutions of the generalized Weierstrass system in $\mathbb{R}^3$, namely (1.1) and (1.2) [**12**]. Let the spinor functions $\psi = (\psi_1, \psi_2)$ and $\varphi = (\varphi_1, \varphi_2)$ be defined in a simply connected domain $W \subset \mathbb{C}$, parametrized by the complex variable z, then the components each satisfy the pair of Dirac equations

$$\mathcal{D}\psi = 0, \qquad \tilde{\mathcal{D}}\varphi = 0 \tag{2.1}$$

where $\mathcal{D}$ and $\tilde{\mathcal{D}}$ are matrix operators which are given by

$$\mathcal{D} = \begin{pmatrix} p & \partial \\ -\bar{\partial} & \bar{p} \end{pmatrix}, \qquad \tilde{\mathcal{D}} = \begin{pmatrix} \bar{p} & \partial \\ -\bar{\partial} & p \end{pmatrix}. \tag{2.2}$$

Before stating how surfaces can be induced from solutions to system (2.1), the following lemma will be useful.

Lemma 1. (a) The components of the spinors ψ and φ which satisfy (2.1) also satisfy the following conditions

$$\partial(\varphi_2\psi_2) + \bar{\partial}(\varphi_1\psi_1) = 0, \quad \bar{\partial}(\psi_1\bar{\varphi}_2) - \partial(\bar{\varphi}_1\psi_2) = 0. \tag{2.3}$$

(b) The one-forms defined by

$$\eta_k = f_k\, dz + \bar{f}_k\, d\bar{z} \qquad k = 1, 2, 3, 4, \tag{2.4}$$

where the coefficients f_k are given by

$$f_1 = \tfrac{i}{2}(\bar{\varphi}_2\bar{\psi}_2 + \varphi_1\psi_1), \qquad f_2 = \tfrac{1}{2}(\bar{\varphi}_2\bar{\psi}_2 - \varphi_1\psi_1),$$
$$f_3 = \tfrac{1}{2}(\bar{\varphi}_2\psi_1 + \varphi_1\bar{\psi}_2), \qquad f_4 = \tfrac{i}{2}(\bar{\varphi}_2\psi_1 - \varphi_1\bar{\psi}_2). \tag{2.5}$$

are closed.

Proof: (a) Expanding the derivatives in (2.3) and substituting (2.1) we obtain that

$$\partial(\varphi_2\psi_2) + \bar{\partial}(\varphi_1\psi_1) = (\partial\varphi_2)\psi_2 + \varphi_2(\partial\psi_2) + (\bar{\partial}\varphi_1)\psi_1 + \varphi_1(\bar{\partial}\psi_1)$$

$$= -\bar{p}\varphi_1\psi_2 - p\varphi_2\psi_1 + p\varphi_2\psi_1 + \bar{p}\varphi_1\psi_2 = 0.$$

The remaining condition is treated the same way.

(b) Consider the case in which $k = 1$, the other cases proceed in a similar way. Upon using the results from part (a), we obtain that

$$d\eta_1 = \bar{\partial} f_1 \, d\bar{z} \wedge dz + \partial \bar{f}_1 \, dz \wedge d\bar{z}$$

$$= \frac{i}{2}(\bar{\partial}(\bar{\varphi}_2\bar{\psi}_2) + \partial(\bar{\varphi}_1\bar{\psi}_1)) \, d\bar{z} \wedge dz + \frac{i}{2}(\partial(\varphi\psi_2) + \bar{\partial}(\varphi_1\psi_1)) d\bar{z} \wedge dz = 0.$$

The next Proposition follows from these.

Proposition 1. Let the spinor functions ψ and φ be defined in a simply connected domain $W \subset \mathbb{C}$ and satisfy the Dirac equations (2.1)-(2.2). Then the one-forms η_k in (2.4) define a surface in $\mathbb{R}^4$ by means of the integrals

$$X^k(z, \bar{z}) = X^k(0) + \int_\Gamma \eta_k, \qquad k = 1, 2, 3, 4. \tag{2.6}$$

The integral in (2.6) is taken over any path Γ in W. By Stokes Theorem and Lemma 1, the integral in (2.6) does not depend on the choice of path. ♣

The induced metric equals

$$e^{2\alpha} \, dz \, d\bar{z} = (|\psi_1|^2 + |\psi_2|^2)(|\varphi_1|^2 + |\varphi_2|^2) \, dz \, d\bar{z} = u_1 u_2 \, dz \, d\bar{z}, \tag{2.7}$$

where $u_1 = |\psi_1|^2 + |\psi_2|^2$ and $u_2 = |\varphi_1|^2 + |\varphi_2|^2$. The mean curvature vector is obtained by calculating

$$\mathbf{H} = \frac{2}{e^{2u}} \mathbf{X}_{z\bar{z}}, \tag{2.8}$$

and the norm of the mean curvature vector is related to p which appears in matrices (2.2) through the expression

$$|p| = \frac{e^{\alpha}}{2}|\mathbf{H}|. \tag{2.9}$$

For $p = \bar{p}$ and $\psi = \pm\varphi$, these formulas reduce to the generalized Weierstrass representation for surfaces in $\mathbb{R}^3$.

In $\mathbb{R}^3$, a single spinor function is sufficient to obtain a surface. In this case the spinor is a lift of the Gauss mapping into nonvanishing spinors, and is required to satisfy a Dirac equation. In $\mathbb{R}^4$, two spinor functions are required to construct a surface, and these functions will form a lift of the Gauss map. In fact, not every lift actually satisfies the Dirac equations (2.1). The lifts which do are defined only up to gauge transformations.

One of the reasons for having this type of formalism available to generate surfaces is that deformations of surfaces can be obtained and studied in a rigorous way. This constitutes a very useful application of these inducing mechanisms. Deformations of a surface are obtained by deforming the potential function p which appears in matrices (2.2) according to some given evolution equation. In particular, we will be interested in considering evolution equations which belong to the DS hierarchy, as these will appear out of the methodology in due course. To this end, let us begin to generate these evolution equations by introducing a corresponding formalism which produces them in a precise way.

To this end, let the operator L be defined as follows

$$L = \begin{pmatrix} 0 & \partial \\ -\bar{\partial} & 0 \end{pmatrix} + \begin{pmatrix} -p & 0 \\ 0 & q \end{pmatrix}. \tag{2.10}$$

We consider deformations of the operator L which take the form of a triple of operators L, A and B. These operators satisfy

$$L_t + [L, A_n] - B_n L = 0. \tag{2.11}$$

Here t will play the role of the evolution parameter. Now for any nonzero spinor Ψ we have

$$[L, \partial_t - A_n]\Psi + B_n L\Psi = L(\partial_t - A_n)\Psi - (\partial_t - A_n)L\Psi + B_n L\Psi$$

$$= L\partial_t\Psi - LA_n\Psi - (\partial_t L)\Psi - L\partial_t\Psi + A_n L\Psi + B_n L\Psi$$

$$= (-\partial_t L - [L, A_n] + B_n L)\Psi.$$

Therefore, we conclude that (2.11) implies that

$$[L, \partial_t - A_n] + B_n L = 0. \tag{2.12}$$

Now if L satisfies (2.11), then the solution of the equation

$$L\Psi = 0 \tag{2.13}$$

is evolved according to the equation

$$\Psi_t = A_n \Psi. \tag{2.14}$$

Theorem 1. For the case in which $n = 1$ such that the matrices A_1 and B_1 in (2.12) are given by

$$A_1 = \begin{pmatrix} \partial & q \\ p & \bar{\partial} \end{pmatrix}, \qquad B_1 = \begin{pmatrix} \bar{\partial} - \partial & -p - q \\ -p - q & \partial - \bar{\partial} \end{pmatrix}, \tag{2.15}$$

then (2.12) is exactly equivalent to the Davey-Stewartson II equations

$$\partial_t p = \partial p + \bar{\partial} p, \qquad \partial_t q = \partial q + \bar{\partial} q. \tag{2.16}$$

Proof: Let $\Psi = (\psi_1, \psi_2)$ be an arbitrary two-component spinor, then by matrix operations we have

$$L\Psi = \begin{pmatrix} \partial\psi_2 - p\psi_1 \\ -\bar{\partial}\psi_1 + q\psi_2 \end{pmatrix}. \tag{2.17}$$

Then based on these matrices, we calculate

$$\begin{aligned}(\partial_t - A_1)L\Psi &= \partial_t \begin{pmatrix} -p\psi_1 + \partial\psi_2 \\ -\bar{\partial}\psi_1 + q\psi_2 \end{pmatrix} - \begin{pmatrix} \partial & q \\ p & \bar{\partial} \end{pmatrix} \begin{pmatrix} -p\psi_1 + \partial\psi_2 \\ -\bar{\partial}\psi_1 + q\psi_2 \end{pmatrix} \\ &= \begin{pmatrix} \partial_t(-p\psi_1 + \partial\psi_2) - \partial(-p\psi_1 + \partial\psi_2) - q(-\bar{\partial}\psi_1 + q\psi_2) \\ \partial_t(-\bar{\partial}\psi_1 + q\psi_2) - p(-p\psi_1 + \partial\psi_2) - \bar{\partial}(-\bar{\partial}\psi_1 + q\psi_2) \end{pmatrix}\end{aligned} \tag{2.18}$$

as well as

$$\begin{aligned}L(\partial_t - A_1)\Psi &= \begin{pmatrix} -p & \partial \\ -\bar{\partial} & q \end{pmatrix} \begin{pmatrix} \partial_t\psi_1 - \partial\psi_1 - q\psi_2 \\ \partial_t\psi_2 - p\psi_1 - \bar{\partial}\psi_2 \end{pmatrix} \\ &= \begin{pmatrix} -p(\partial_t\psi_1 - \partial\psi_1 - q\psi_2) + \partial(\partial_t\psi_2 - p\psi_1 - \bar{\partial}\psi_2) \\ -\bar{\partial}(\partial_t\psi_1 - \partial\psi_1 - q\psi_2) + q(\partial_t\psi_2 - p\psi_1 - \bar{\partial}\psi_2) \end{pmatrix}\end{aligned} \tag{2.19}$$

with

$$B_1 L\Psi = \begin{pmatrix} \bar{\partial} - \partial & -p - q \\ -p - q & \partial - \bar{\partial} \end{pmatrix} \begin{pmatrix} -p\psi_1 + \partial\psi_2 \\ -\bar{\partial}\psi_1 + q\psi_2 \end{pmatrix}$$

$$= \begin{pmatrix} (\bar{\partial} - \partial)(-p\psi_1 + \partial\psi_2) - (p+q)(-\bar{\partial}\psi_1 + q\psi_2) \\ -(p+q)(-p\psi_1 + \partial\psi_2) + (\partial - \bar{\partial})(-\bar{\partial}\psi_1 + q\psi_2) \end{pmatrix}. \tag{2.20}$$

Substituting (2.18), (2.19) and (2.20) into (2.12) and simplifying, the top element of the resulting matrix reduces to

$$\psi_1(\partial_t p - \partial p - \bar{\partial} p) = 0,$$

and the lower element of the matrix reduces to

$$\psi_2(-\partial_t q + \partial q + \bar{\partial} q) = 0.$$

These results are exactly system (2.16). ♣

These calculations can be best carried out by means of symbolic manipulation. In fact, for the cases $n = 2$ and $n = 3$, the basic structure of the matrices will be obtained, and then the rest of the proof makes use of this **[11]**.

It should be noted that the DS I hierarchy is a related system of nonlinear equations which are obtained from the DS II hierarchy by replacing the variables z and $\bar{z}$ by real-valued variables x and y.

Consider the following reduction of the system (2.16) which is specified by taking

$$p = -u, \qquad q = \bar{u}. \tag{2.21}$$

It is then seen that system (2.16) is compatible under these substitutions and the pair reduces to the single expression

$$\partial_t u = \partial u + \bar{\partial} u. \tag{2.22}$$

Theorem 2. For the case in which $n = 2$, the matrices

$$A_2 = \begin{pmatrix} -\partial^2 - v_1 & q\bar{\partial} - \bar{\partial} q \\ -p\partial + \partial p & \bar{\partial}^2 + v_2 \end{pmatrix} \tag{2.23}$$

and

$$B_2 = \begin{pmatrix} \partial^2 + \bar{\partial}^2 + v_1 + v_2 & -(p+q)\bar{\partial} + \bar{\partial} q - 2\bar{\partial} p \\ (p+q)\partial - \partial p + 2\partial q & -(\partial^2 + \bar{\partial}^2) - (v_1 + v_2) \end{pmatrix} \tag{2.24}$$

where v_1 and v_2 satisfy

$$\bar{\partial} v_1 = -2\partial(pq), \qquad \partial v_2 = -2\bar{\partial}(pq) \tag{2.25}$$

generate the following system of equations under (2.12)

$$\partial_t p = \partial^2 p + \bar{\partial}^2 p + (v_1 + v_2)p, \qquad \partial_t q = -\partial^2 q - \bar{\partial}^2 q - (v_1 + v_2)q. \tag{2.26}$$

Proof: With $L\Psi$ given by (2.17), we have that

$$(\partial_t - A_2)L\Psi = \begin{pmatrix} \partial_t (L\Psi)_1 + \partial^2 (L\Psi)_1 + v_1 (L\Psi)_1 - q\bar{\partial}(L\Psi)_2 + (\bar{\partial} q)(L\Psi)_2 \\ \partial_t (L\Psi)_2 + p\partial (L\Psi)_1 - \partial p (L\Psi)_1 - \bar{\partial}^2 (L\Psi)_2 - v_2 (L\Psi)_2 \end{pmatrix} \tag{2.27}$$

and

$$(\partial_t - A_2)\Psi = \begin{pmatrix} \partial_t \psi_1 + \partial^2 \psi_1 + v_1 \psi_1 - q\bar{\partial}\psi_2 + \bar{\partial} q \psi_2 \\ \partial_t \psi_2 + p\partial \psi_1 - \partial p \psi_1 - \bar{\partial}^2 \psi_2 - v_2 \psi_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 \\ \Lambda_2 \end{pmatrix} \tag{2.28}$$

$$L(\partial_t - A_2)\Psi = \begin{pmatrix} -p\Lambda_1 + \partial \Lambda_2 \\ -\bar{\partial}\Lambda_1 + q\Lambda_2 \end{pmatrix}. \tag{2.29}$$

Substituting (2.27), (2.28) and (2.29) into (2.12), the pair of equations in (2.26) is generated by calculation, with the p equation the upper component and the q equation as the lower component. ♣

If we consider the reduction given in (2.21), the q equation in (2.26) is not compatible with the p equation. However, by modifying the pair of matrices in a straightforward way, it is possible to obtain a compatible pair from (2.12). The following result formalizes this objective.

Corollary 1. If the matrices (A_2, B_2) in Theorem 2 are replaced by the matrices (iA_2, iB_2), where A_2 and B_2 are given by (2.23) and (2.24) respectively, then (2.12) generates the following pair of equations

$$p_t = i(\partial^2 p + \bar{\partial}^2 p + (v_1 + v_2)p), \qquad q_t = -i(\partial^2 q + \bar{\partial}^2 q + (v_1 + v_2)q). \tag{2.30}$$

Given Corollary 1, equations (2.30) are compatible under the reduction given in (2.21), which reduces the pair of equations given in (2.30) to the single equation

$$\partial_t u = i(\partial^2 u + \bar{\partial}^2 u + (v + \bar{v})u), \qquad \bar{\partial} v = \partial(|u|^2). \tag{2.31}$$

The last system in the hierarchy to be considered here is presented in the next result.

Theorem 3. For the case in which $n = 3$, define the matrices

$$A_3 = \begin{pmatrix} \partial^3 + \frac{3}{2}v_2\partial - 3w_1 & q\bar{\partial}^2 - (\bar{\partial}q)\bar{\partial} + \bar{\partial}^2 q + \frac{3}{2}v_2 q \\ p\partial^2 - (\partial p)\partial + \partial^2 p + \frac{3}{2}v_1 p & \bar{\partial}^3 + \frac{3}{2}v_2\bar{\partial} - 3w_2 \end{pmatrix} \tag{2.32}$$

and

$$B_3 = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \tag{2.33}$$

where

$$\begin{aligned} b_{11} = -b_{22} &= \bar{\partial}^3 - \partial^3 - \frac{3}{2}(v_1\partial - v_2\bar{\partial}) + 3(w_1 - w_2), \\ b_{12} &= -(p+q)\bar{\partial}^2 - \frac{3}{2}(p+q)v_2 - (3\bar{\partial}p - \bar{\partial}q)\bar{\partial} - (3\bar{\partial}^2 p + \bar{\partial}^2 q), \\ b_{21} &= -(p+q)\partial^2 - \frac{3}{2}(p+q)v_1 - (3\partial q - \partial p)\partial - (3\partial^2 q + \partial^2 p). \end{aligned} \tag{2.34}$$

such that the v_i and w_i satisfy

$$\bar{\partial}v_1 = -2\partial(pq), \qquad \partial v_2 = -2\bar{\partial}(pq), \qquad \bar{\partial}w_1 = \partial(p\partial q), \qquad \partial w_2 = \bar{\partial}(q\bar{\partial}p). \tag{2.35}$$

Then (2.12) reduces to the following pair of equations in terms of p and q,

$$\begin{aligned} \partial_t p &= \partial^3 p + \bar{\partial}^3 p + \tfrac{3}{2}(v_1\partial p + v_2\bar{\partial}p) + 3(w_1 - w_2 + \tfrac{1}{2}\partial v_1)p, \\ \partial_t q &= \partial^3 q + \bar{\partial}^3 q + \tfrac{3}{2}(v_1\partial q + v_2\bar{\partial}q) - 3(w_1 - w_2 - \tfrac{1}{2}\partial v_2)q. \end{aligned} \tag{2.36}$$

This system can be put in the following form by redefining v_i

$$\begin{aligned} \partial_t p &= \partial^3 p + \bar{\partial}p^3 + 3(v_1\partial p + v_2\bar{\partial}p) - 3(\partial^{-1}[(q\bar{\partial}p)_{\bar{z}}] + \bar{\partial}^{-1}[(q\partial p)_z])p, \\ \partial_t q &= \partial^3 q + \bar{\partial}^3 q + 3(v_1\partial q + v_2\bar{\partial}q) - 3(\partial^{-1}[(p\bar{\partial}q)_{\bar{z}}] + \bar{\partial}^{-1}[(p\partial q)_z])q \end{aligned} \tag{2.37}$$

The equations in (2.37) are compatible under (2.21) and cause (2.37) to reduce to the single equation

$$\partial_t u = \partial^3 u + \bar{\partial}^3 u + 3(v\partial u + \bar{v}\bar{\partial}u) + 3(w + w')u,$$

$$\bar{\partial}v = \partial(|u|^2), \qquad \bar{\partial}w = \partial(\bar{u}\partial u), \qquad \partial w' = \bar{\partial}(\bar{u}\bar{\partial}u). \tag{2.38}$$

Frequently, (2.31) and (2.38) are referred to as the DS_2 and DS_3 equations, respectively. In fact, (2.38) is also compatible with the additional constraint $u = \bar{u}$, and reduces to the modified

Novikov-Veselov equation. In fact, since $\bar{\partial}v = \partial(u^2)$ and $\bar{\partial}w = \frac{1}{2}\partial(\partial u^2) = \frac{1}{2}\partial(\bar{\partial}v) = \frac{1}{2}\bar{\partial}(\partial v)$ it follows that $w = \frac{1}{2}\partial v$ and $w' = \frac{1}{2}\bar{\partial}v$. Then (2.38) is given by

$$\partial_t u = \partial^3 u + \bar{\partial}^3 u + 3(v\partial u + \bar{v}\bar{\partial}u) + \frac{3}{2}(\partial v + \bar{\partial}\bar{v})u, \quad \bar{\partial}v = \partial(u^2). \tag{2.39}$$

This is exactly the Novikov-Veselov equation.

3. GAUSS MAP, SURFACES AND DEFORMATIONS OF SURFACES IN $\mathbb{R}^4$.

The Grassmannian of oriented two-planes in $\mathbb{R}^n$ is modeled by the quadric Q_{n-2} in CP^{n-1} defined by the equation

$$\sum_{k=1}^{n} z_k^2 = 0. \tag{3.1}$$

For any point $P = (z_1, \cdots, z_n)$ on Q_{n-2} if $z_k = a_k + ib_k$, we obtain a pair of real vectors $A = (a_1, \cdots, a_n)$ and $B = (b_1, \cdots, b_n)$ which satisfy

$$|A| = |B|, \quad A \cdot B = 0. \tag{3.2}$$

These equations are equivalent to (3.1), and A and B cannot be zero since homogeneous coordinates $(z_1, \cdots, z_n)$ of a point in CP^{n-1} are not all zero. This pair A, B form an orthogonal basis of an oriented two plane Π. In $\mathbb{R}^4$, we give some facts concerning $Q_2 \subset \mathbb{C}P^3$. There are some useful connections between algebraic and differential geometry which are outlined here **[13,14]**. The map σ from $\mathbb{C} \times \mathbb{C}$ given by

$$\sigma(w_1, w_2) \to (1 + w_1 w_2, i(1 - w_1 w_2), w_1 - w_2, -i(w_1 + w_2)), \tag{3.3}$$

has the property that $\sigma^2 = \sum_{k=1}^4 \sigma_k^2 = 0$. Hence $[\sigma]$ takes values in $Q_2 = \{[Z] \in \mathbb{C}P^3 | Z^2 = 0\}$. Moreover, on $\sigma(\mathbb{C} \times \mathbb{C})$, the related mapping σ^{-1} is given by

$$(z_1, z_2, z_3, z_4) \to (w_1, w_2) = (\frac{z_3 + iz_4}{z_1 - iz_2}, \frac{-z_3 + iz_4}{z_1 - iz_2}). \tag{3.4}$$

Thus, $[\sigma]$ is a biholomorphic map from $\mathbb{C} \times \mathbb{C}$ into Q_2, which when $(w_1, w_2) \in \mathbb{C} \times \mathbb{C}$ are considered as homogeneous coordinates on $\mathbb{C}P^1 \times \mathbb{C}P^1$. It extends to a biholomorphic map of $\mathbb{C}P^1 \times \mathbb{C}P^1$

onto Q_2. If $\mathbb{C}P^3$ is considered under the Fubini-Study metric of constant holomorphic curvature two, the induced metric on Q_2, expressed in terms of (w_1, w_2) has the form

$$ds^2 = \frac{2|dw_1|^2}{(1+|w_1|^2)^2} + \frac{2|dw_2|^2}{(1+|w_2|^2)^2}. \tag{3.5}$$

This implies that Q_2 is the product of two standard spheres of constant Gauss curvature of two.

An oriented two-plane in $\mathbb{R}^4$ is defined by a positively oriented orthonormal basis $e_1 = (e_{1,1}, \cdots, e_{1,4})$ and $e_2 = (e_{2,1}, \cdots, e_{2,4})$ defined up to rotations. There exists a one-to-one correspondence between components of the e_i and points of the quadric $\mathbb{Q} \subset \mathbb{C}P^3$ defined by (3.1) as

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0. \tag{3.6}$$

The correspondence is given by $z_k = e_{1,k} + ie_{2,k}$ for $k = 1, \cdots, 4$. Consider the following change of coordinates from y_i to z_i

$$z_1 = \frac{i}{2}(y_1 + y_2), \qquad z_2 = \frac{1}{2}(y_1 - y_2), \qquad z_3 = \frac{1}{2}(y_3 + y_4), \qquad z_4 = \frac{i}{2}(y_3 - y_4). \tag{3.7}$$

The quadric in terms of the y_i can then be written

$$y_1 y_2 = y_3 y_4.$$

This establishes a correspondence between the space $\mathbb{C}P^1 \times \mathbb{C}P^1$ and $\tilde{G}_{4,2}$ in the form of a product

$$\tilde{G}_{4,2} = \mathbb{C}P^1 \times \mathbb{C}P^1. \tag{3.8}$$

It is this equivalence which allows us to decompose the Gauss map into a pair of maps which can be written as

$$G = (G_\psi, G_\varphi), \quad G : W \to \tilde{G}_{4,2} \quad Q \in W \to (X_z^1(Q), X_z^2(Q), X_z^3(Q), X_z^4(Q)),$$

and G_ψ and G_φ can be represented in terms of a pair of spinors

$$G_\psi = (\psi_1, \psi_2) \in \mathbb{C}P^1, \qquad G_\varphi = (\varphi_1, \varphi_2) \in \mathbb{C}P^1. \tag{3.10}$$

The actual coordinates of the surface can be written in terms of the $(\psi_i\ ,\ \varphi_i)$ presented in (2.6).

Such a decomposition will however not be unique. In fact, the spinors ψ and φ will be defined only up to a gauge transformation [**7-6**], that is, a transformation of the form,

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \to \begin{pmatrix} e^f \psi_1 \\ e^{\bar{f}} \psi_2 \end{pmatrix}, \qquad \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} \to \begin{pmatrix} e^{-f} \varphi_1 \\ e^{-\bar{f}} \varphi_2 \end{pmatrix}. \tag{3.11}$$

Define the functions κ_α and τ_α in terms of ψ_α and φ_α, as follows

$$\begin{array}{ll} \kappa_1 = e^f \psi_1, & \tau_1 = e^{-f} \varphi_1 \\ \kappa_2 = e^{\bar{f}} \psi_2, & \tau_2 = e^{-\bar{f}} \varphi_2. \end{array} \tag{3.12}$$

It is clear that the ratios $\kappa_1/\bar{\kappa}_2$ and $\tau_1/\bar{\tau}_2$ are left invariant by the gauge transformation given by (3.11).

Suppose, for example, a lift is constructed which is based on the initial pair of functions $(s_1, s_2) = (e^{i\theta} \cos\eta, \sin\eta)$. Then a pair of functions (ψ_1, ψ_2) can be sought such that the ratio is preserved, namely, $\psi_1/\bar{\psi}_2 = s_1/s_2$ and (ψ_1, ψ_2) satisfy (2.1). From (3.11), we can write

$$\psi_1 = e^f s_1, \qquad \psi_2 = e^{\bar{f}} s_2. \tag{3.13}$$

Let us obtain equations satisfied by the functions in the lift as well as f. Requiring that the ψ_α in (3.13) satisfy Dirac equations (2.1), this will be the case provided that (θ, η) and f satisfy the equations

$$\partial(e^{\bar{f}} \sin\eta) + p e^{f+i\theta} \cos\eta = 0, \qquad \bar{\partial}(e^{f+i\theta} \cos\eta) = \bar{p} e^{\bar{f}} \sin\eta. \tag{3.14}$$

Eliminating the function p from the equations in (3.14), we obtain

$$\bar{\partial} f + i(\bar{\partial}\theta) \cos^2\eta = 0.$$

Using (3.14) to obtain $\partial \bar{f}$, we obtain an expression for p

$$p = -e^{\bar{f} - f - i\theta} (i\partial\theta \sin\eta \cos\eta + \partial\eta).$$

Once the ψ_α have been fully determined, the conditions (2.3) can be used to calculate the components of the second lift G_φ, since these components must satisfy the remaining equation in (2.1).

In fact, for a certain class of functions f, the entire system (2.1) may be preserved in form, or may be said to be gauge invariant.

Proposition 2. If the gauge function f in gauge transformation (3.11) satisfies $\bar{\partial} f = 0$, then system (2.1) is left invariant under (3.11) provided that the potential function p in (2.2) is transformed or gauged according to $P = pe^{\bar{f}-f}$.

Proof: Suppose the functions ψ_α and φ_α satisfy system (2.1). Differentiating κ_1 in (3.12) with respect to $\bar{z}$, we obtain

$$\bar{\partial}\kappa_1 = (\bar{\partial} f)e^f \psi_1 + e^f \bar{\partial}\psi_1 = e^f (\bar{\partial} f)\psi_1 + \bar{p}e^f \psi_2 = \bar{P}\kappa_2.$$

A similar result applies to κ_2. Next differentiating τ_1 we have

$$\bar{\partial}\tau_1 = -(\bar{\partial} f)e^{-f}\varphi_1 + e^{-f}\bar{\partial}\varphi_1 = -e^{-f}(\bar{\partial} f)\varphi_1 + pe^{-f}\varphi_2 = P\tau_2.$$

A similar result holds for the function τ_2. ♣

Consider now deformations of surfaces. Konopelchenko **[2]** introduced the definition of the DS deformations of a surface. Integrable deformations of surfaces generated by the Weierstrass formulas will be constructed. As mentioned earlier, this is one of the main applications of the generalized Weierstrass representation, as it gives a way to construct integrable deformations of immersed surfaces.

Let us start with surfaces in $\mathbb{R}^3$. Deformations of the functions ψ and φ are considered such that there are differential operators A_n, B_n, C_n and D_n such that

$$\psi_{t_n} = A_n\psi + B_n\varphi, \qquad \varphi_{t_n} = C_n\psi + D_n\varphi. \tag{3.15}$$

For given operators, the compatibility condition of (3.15) with (1.1) gives a nonlinear partial differential equation for p. Changing the operators on the right of (3.15) generates an infinite hierarchy of integrable equations for p. The deformations of ψ and φ described by (3.15) generates the corresponding deformations of the corresponding coordinates $X^i(z,\bar{z},t)$. For example, when $n = 1$, the operators in (3.15) can be written as

$$A_1 = \alpha\bar{\partial}, \quad B_1 = \gamma p, \quad C_1 = \alpha q, \quad D_1 = \gamma\partial, \tag{3.16}$$

and system (3.15) turns out to be linear. Equations corresponding to higher values of n are nonlinear equations, such as the ones presented in Theorems 2 and 3.

Now to generate integrable deformations of surfaces immersed in $\mathbb{R}^4$, it is assumed that the components of both spinor solutions to (2.1) and (2.2) evolve with parameter t according to (3.15) under the same operators A_n, B_n, C_n and D_n. The compatibility conditions for (2.1) with (3.15) fix the dependence of ψ and φ as well as p and q on the parameter t, and consequently define the deformations of the corresponding surfaces. Thus, the coordinates X^i for $i = 1, 2, 3, 4$ of the given surface which are calculated by means of (2.6) are defined in terms of t as well. Specific cases will be governed by different reductions of the hierarchy. This can now be put together and stated in a precise way. First, let us note that A_n in Theorems 1 to 3 depend on two functional parameters p and q and for the case in which $p = -u$ and $q = \bar{u}$, define $A_n^+ = A_n$. For the alternate case, $p = -\bar{u}$ and $q = u$, set $A_n^- = A_n$. In terms of these new operators, the following Proposition can be stated.

Proposition 3: Let surface Σ be defined by (2.5) and (2.6) for certain initial spinors ψ^0 and φ^0 which satisfy (2.1), and let $p(z, \bar{z}, t)$ be a deformation of the potential whose evolution is described by the reduced equations (2.16), (2.31) or (2.38). Then the equations (2.5) and (2.6) combined with a pair of the following equations

$$\begin{aligned} \psi_t &= A_1^+ \psi, \qquad \varphi_t = A_1^- \varphi, \\ \psi_t &= i A_2^+ \psi, \qquad \varphi_t = -i A_2^- \varphi, \\ \psi_t &= A_3^+ \psi, \qquad \varphi_t = A_3^- \varphi, \end{aligned} \tag{3.17}$$

such that $\psi_{t=0} = \psi^0$ and $\varphi_{t=0} = \varphi^0$ define deformations of the surface which is governed by the reduced evolution equations (2.16), (2.31) or (2.38), respectively.

Proof: The deformation of p in (2.2) is described by (2.11)-(2.12), so the spinors ψ^0 and φ^0, which satisfy system (2.1), are deformed according to (3.17). Thus, for any t, the resulting spinors still satisfy the Dirac equations (2.1). From Proposition 1, however, solutions to this system define a new related surface Σ_t in $\mathbb{R}^4$ by means of Weierstrass equations (2.4)-(2.6). This process has generated a deformed surface Σ_t such that Σ_0 coincides with the original surface generated by spinors (ψ^0, φ^0). ♣

Finally several results related to preservation of surface structure under deformation will be

given.

Proposition 4. Let h be any of the functions $\bar{\psi}_1\bar{\varphi}_1$, $\bar{\psi}_1\varphi_2$, $\psi_2\bar{\varphi}_1$, $\psi_2\varphi_2$ which appear in the conservation laws (2.3). Then with respect to DS_2 equation (2.31), each of the functionals

$$J(h) = \int_\Sigma h\, dz \wedge d\bar{z} \tag{3.18}$$

is conserved with respect to the evolution parameter, $\partial_t J = 0$.

Proof: The claim will be shown for the case in which $h = \bar{\psi}_1\bar{\varphi}_1$, the others follow similarly. To do this, the evolution equations for $\bar{\psi}_1$ and $\bar{\varphi}_1$ are required. These are given by the second pair of matrix equations in (3.17) such that $A_2^{\pm}$ are based on the A_2 given in (2.23). These equations are given by

$$\bar{\psi}_{1t} = -i[(-\bar{\partial}^2 - \bar{v})\bar{\psi}_1 + (u\partial - \partial u)\bar{\psi}_2], \qquad \bar{\varphi}_{1t} = i[-(\bar{\partial}^2 + \bar{v})\bar{\varphi}_1 + (\bar{u}\partial - \partial\bar{u})\bar{\varphi}_2]. \tag{3.19}$$

Let

$$J = \int_\Sigma \bar{\psi}_1\bar{\varphi}_1\, dz \wedge d\bar{z},$$

then differentiation proceeds through the integral to give

$$\partial_t J = \int_\Sigma (\bar{\psi}_{1t}\bar{\varphi}_1 + \bar{\psi}_1\bar{\varphi}_{1t})\, dz \wedge d\bar{z}$$

$$= -i\int_\Sigma [(-\bar{\partial}^2\bar{\psi}_1 - \bar{v}\bar{\psi}_1)\bar{\varphi}_1 + (u\partial\bar{\psi}_2 - (\partial u)\bar{\psi}_2)\bar{\varphi}_1 + \bar{\psi}_1(\bar{\partial}^2\bar{\varphi}_1 + \bar{v}\bar{\varphi}_1) - \bar{\psi}_1(\bar{u}\partial\bar{\varphi}_2 - (\partial\bar{u})\bar{\varphi}_2)]\, dz \wedge d\bar{z}$$

$$= -i\int_\Sigma [(u\partial\bar{\psi}_2 - (\partial u)\bar{\psi}_2)\bar{\varphi}_1 - (\bar{u}\partial\bar{\varphi}_2 - (\partial\bar{u})\bar{\varphi}_2)\bar{\psi}_1]\, dz \wedge d\bar{z},$$

where (3.19) and integration by parts has been used to simplify this. Integrating by parts once more, we obtain

$$\partial_t J = -i\int_\Sigma [u\bar{\varphi}_1\partial\bar{\psi}_2 + u\partial(\bar{\varphi}_1\bar{\psi}_2) - \bar{u}\partial(\bar{\varphi}_2\bar{\psi}_1) - \bar{u}\bar{\psi}_1\partial\bar{\varphi}_2]\, dz \wedge d\bar{z}$$

$$= -i\int_\Sigma [2u\bar{\varphi}_1\partial\bar{\psi}_2 + u\bar{\psi}_2\partial\bar{\varphi}_1 - 2\bar{u}\bar{\psi}_1\partial\bar{\varphi}_2 - \bar{u}\bar{\varphi}_2\partial\bar{\psi}_1]\, dz \wedge d\bar{z}.$$

Finally, replace both u and $\bar{u}$ in this by the corresponding derivatives from (2.1) to get

$$\partial_t J = -i\int_\Sigma [2\bar{\partial}\bar{\varphi}_2\partial\bar{\psi}_2 + \partial\bar{\psi}_1\partial\bar{\varphi}_1 - 2\bar{\partial}\psi_2\partial\bar{\varphi}_2 - \partial\bar{\varphi}_1\partial\bar{\psi}_1]\, dz \wedge d\bar{z}$$

$$= -2i \int_\Sigma [\bar{\partial}\bar{\varphi}_2 \partial\bar{\psi}_2 - \bar{\partial}\bar{\psi}_2 \partial\bar{\varphi}_2] \, dz \wedge d\bar{z}$$

$$= -2i \int_\Sigma [-\bar{\varphi}_2 \bar{\partial}\partial\bar{\psi}_2 + \bar{\varphi}_2 \partial\bar{\partial}\bar{\psi}_2] \, dz \wedge d\bar{z} = 0.$$

Proposition 5. Let W be the Willmore functional defined as

$$W = \int_\Sigma |u|^2 \, dz \wedge d\bar{z}. \tag{3.20}$$

Then the DS_2 deformation (2.31) of tori preserves the Willmore functional defined by (3.20).

Proof: Differentiating W given in (3.20) with respect to t through the integral , we obtain

$$\partial_t W = \int_\Sigma (u_t \bar{u} + u \bar{u}_t) \, dz \wedge d\bar{z}.$$

Replacing the derivatives with respect to t by the DS_2 equation (2.31), this becomes

$$\partial_t W = i \int_\Sigma ((\partial^2 u + \bar{\partial}^2 u + 2(v + \bar{v})u)\bar{u} - u(\bar{\partial}^2 \bar{u} + \partial^2 u + 2(v + \bar{v})\bar{u})) \, dz \wedge d\bar{z}$$

$$= i \int_\Sigma (\bar{u}\partial^2 u - u\partial^2 \bar{u} + \bar{u}\bar{\partial}^2 u - u\bar{\partial}^2 \bar{u}) \, dz \wedge d\bar{z}.$$

Finally, integrating this by parts twice, the required result is obtained

$$\partial_t W = i \int_\Sigma (\bar{u}\partial^2 u - (\partial^2 u)\bar{u} + \bar{u}\bar{\partial}^2 u - (\bar{\partial}^2 u)\bar{u}) \, dz \wedge d\bar{z} = 0.$$

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